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SOME SINGULAR SOLUTIONS OF THE THEORY OF ELASTICITY†

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Singular solutions corresponding to "double forces without moment", an "extension-compression centre", and "double forces with moment", obtained previously for an isotropic medium [2, 3] by differentiation of the Kelvin fundamental solution, are constructed for media with an arbitrary elastic anisotropy using the method of multiple expansions [1]. © 1997 Elsevier Science Ltd. All rights reserved.

1. FUNDAMENTAL RELATIONS

Consider an anisotropic elastic medium, the equilibrium equations of which can be written in the form

$$\mathbf{A}(\partial_x)\mathbf{u} \equiv -\operatorname{div}_x \mathbf{C} \cdot\cdot \nabla_x \mathbf{u} = 0 \tag{1.1}$$

where **A** is a matrix differential operator of the equilibrium equations, **u** is the displacement vector, and **C** is a fourth-order elasticity tensor. We will assume that the medium is hyperelastic while the tensor **C** is strictly elliptic.

The fundamental solutions of Eqs (1.1) in R^3 in closed form are known only for certain particular forms of elastic anisotropy [4]. When the medium has arbitrary anisotropy, either the method of expansion in plane waves [5] or the method of multipole expansions [1] are used to construct the fundamental solutions. In both cases an integral Fourier transformation is employed which enables the Fourier transformed fundamental solution (symbol) to be written in the form

$$\mathbf{E}^\wedge(\xi) = \mathbf{A}^\wedge(\xi)^{-1} \tag{1.2}$$

where $\mathbf{A}^\wedge(\xi) \equiv (2\pi)^2\xi \cdot \mathbf{C} \cdot \xi$ is the symbol of the differential operator of Eqs (1.1). Formula (1.2) shows that the symbol $\mathbf{E}^\wedge$ is positively homogeneous of power –2 and strictly elliptic, in view of the strict ellipticity of the tensor **C**.

For the Fourier transformation of expression (1.2) and to construct a truly fundamental solution, the symbol $\mathbf{E}^\wedge$ is expanded in series in multipoles (a series in surface spherical harmonics) [1]

$$\mathbf{E}^\wedge(\xi) = |\xi|^{-2} \sum_{n=0,2,4,\ldots} \sum_k \mathbf{E}_{nk} Y_n^k(\xi'), \quad \xi' = \frac{\xi}{|\xi|} \tag{1.3}$$

where E_{nk} are matrix coefficients defined by integration over a sphere of unit radius, and Y_n^k are surface spherical harmonics, and summation over k is everywhere carried out from $k = 1$ to $k = 2n + 1$. However, we use the formula given in [6], which defines the Fourier transformation of the symbols of integral operators with a weak (integrable) singularity, which enables the required fundamental solution to be represented also in the form of a multipole series

$$\mathbf{E}(\mathbf{x}) = |\mathbf{x}|^{-1} \sum_{n=0,2,4,\ldots} \gamma_n \sum_k \mathbf{E}_{nk} Y_n^k(\mathbf{x}'), \quad \mathbf{x}' = \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{1.4}$$

$$\gamma_n = (-1)^n \pi^{1/2} \Gamma\left(\frac{n+1}{2}\right) \Big/ \Gamma\left(\frac{n+2}{2}\right)$$

(γ_n are the transition coefficients of transform space to the initial space). Questions of the convergence of expansions of the form (1.3) and (1.4) are considered in [1].

2. SINGULAR SOLUTIONS

A double force without moment. Consider a system of two forces, a distance h from one another and acting along

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along a single straight line in opposite directions. Each of the forces has a magnitude of P/h. Bearing in mind the fact that the displacement field of a unit force in an unbounded medium is governed by the fundamental solution E, and assuming that the parameter h tends to zero, we obtain

$$\mathbf{u} = \nabla_n \mathbf{E} \cdot \mathbf{n} P \tag{2.1}$$

($\mathbf{n}$ is the vector of unit length directed along the line of action of the forces).

It is extremely difficult to differentiate Eq. (1.4) directly to obtain the displacement field $\mathbf{u}$ from (2.1). The situation can be simplified if we carry out a Fourier transformation of both sides in (2.1)

$$\mathbf{u}^\wedge(\xi) = \mathbf{n} \cdot \mathbf{V}^\wedge(\xi) \cdot \mathbf{n} P, \quad \mathbf{V}^\wedge(\xi) = 2\pi i \xi \otimes \mathbf{E}^\wedge(\xi) \tag{2.2}$$

It can be seen that the (rank 3) tensor symbol $\mathbf{V}^\wedge$ is positively homogeneous with respect to ξ of power -1. The further calculations involved in the method of multipole expansions are discussed below.

An extension-compression centre. This is the so-called set of three "double forces without moment", which act along three mutually orthogonal directions. Here the displacement field in the medium is given by the expression

$$\mathbf{u} = \operatorname{div}(\mathbf{E}) P \tag{2.3}$$

The Fourier transformation of both sides of (2.3) can also be represented in terms of the symbol $\mathbf{V}^\wedge$

$$\mathbf{u}^\wedge(\xi) = \operatorname{tr} \mathbf{V}^\wedge(\xi) P \equiv V_{ij}^{\wedge j}(\xi) P \tag{2.4}$$

In (2.4) we have used the rule of summation over repeated subscripts.

A double force with moment. This case differs from the case of a double force without a moment in the fact that here the oppositely directed forces form a pair of forces. The displacement field in this case has the form

$$\mathbf{u} = \nabla_n \mathbf{E} \cdot \mathbf{n}_\perp P \tag{2.5}$$

where $\mathbf{n}$ is the vector of unit length, defining the direction of the "arm" of the pair, whereas $\mathbf{n}$ is the affine vector, which indicates the direction in which one of the forces acts. Formula (2.5) can also take a skew-symmetric form

$$\bar{\mathbf{u}} = (\nabla_n \mathbf{E} \cdot \mathbf{n}_\perp - \nabla_{n_\perp} \mathbf{E} \cdot \mathbf{n}) P / 2 \tag{2.6}$$

A Fourier transformation of (2.6) gives

$$\mathbf{u}^\wedge(\xi) = (\mathbf{n} \cdot \mathbf{V}^\wedge(\xi) \cdot \mathbf{n}_\perp - \mathbf{n}_\perp \cdot \mathbf{V}^\wedge(\xi) \cdot \mathbf{n}) P / 2 \tag{2.7}$$

Sometimes this case of loading is called a "centre of rotation around the $n \times n$ axis" [7].

3. THE DISPLACEMENT FIELDS

To determine the displacement fields we will expand the symbol $\mathbf{V}^\wedge$ in a multipole series

$$\mathbf{V}^\wedge(\xi) = |\xi|^{-1} \sum_{n=1,3,5,\ldots} \sum_k \mathbf{V}_{nk} Y_n^k(\xi') \tag{3.1}$$

By analogy with (1.3) the (rank 3) tensor coefficients $\mathbf{V}_{mk}$ are found by integrating the symbol $\mathbf{V}^\wedge$ over the unit sphere S

$$\mathbf{V}_{nk} = \int_S \mathbf{V}^\wedge(\xi') Y_n^k(\xi') d\xi'$$

The Fourier transform of the multipole series (3.1) is quickly obtained from (3.1) by introducing the Bochner coefficients γ_n

$$\mathbf{V}(\mathbf{x}) = |\mathbf{x}|^{-2} \sum_{n=1,3,5,\ldots} \gamma_n \sum_k \mathbf{V}_{nk} Y_n^k(\mathbf{x}')$$
$$\gamma_n = i^n \pi^{-1/2} \Gamma\left(\frac{n+2}{2}\right) \Big/ \Gamma\left(\frac{n+1}{2}\right) \tag{3.2}$$

Here, in view of (2.2) the summation in (3.2) is essentially carried out over spherical harmonics of odd powers.

In conclusion it remains to convolute the operator $\mathbf{V}$ and multiply by P. Hence, using the multiple expansion of the symbol $\mathbf{V}^\wedge$ we can construct the displacement fields in an anisotropic medium from the action of the fundamental singular forces.

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